Unit 2:
The Antiderivative

Definition 2.1. $F(x)$ is said to be *an antiderivative* of $f(x)$ on an interval if $F'(x) = f(x)$ or equivalently $\frac{d}{dx}(F(x)) = f(x)$ for every value of x on the interval.

Example 1. Find an antiderivative of x^3

Solution: The exponent on x is 3, so in searching for an antiderivative (thinking of power rule in reverse) it makes sense to begin our search with x^4. Testing this hypothesis we see that

$$\frac{d}{dx}(x^4) = 4x^3$$

We require x^3 which we observe to be $\frac{1}{4}(4x^3)$. With this in mind we see that

$$\frac{d}{dx}(\frac{1}{4}(x^4)) = x^3$$

Definition 2.2. Notice that we defined *an* antiderivative as opposed to *the* antiderivative of a function. The reason for this can be demonstrated quite easily as follows:

First we see for example that

$$\frac{d}{dx}(\frac{1}{4}(x^4)) = x^3$$

and that

$$\frac{d}{dx}(\frac{1}{4}(x^4) + 3) = x^3 + 0 = x^3$$

and that in general

$$\frac{d}{dx}(\frac{1}{4}(x^4) + C) = x^3, \quad \textit{where C is any arbitrary constant}$$

For this reason we now define *the general antiderivative* of a function $f(x)$ to be the sum of an antiderivative of $f(x)$ and an arbitrary constant C.

Example 2. Find the general antiderivative, $F(x)$, of $f(x) = x^4$

Solution: After some thought (see example 1) we see that $\frac{1}{5}(x^5)$ is an antiderivative of x^4. In light of definition 2 we then have

$$F(x) = \frac{1}{5}(x^5) + C$$

is the general antiderivative of x^4.

We now introduce some general rules and notation:

The general antiderivative of $f(x)$ will be denoted

$$\int f(x)dx$$

where the sybol $\int$ is called the *integral* sign, $f(x)dx$ is called the *integrand*, and the presence of dx signifies that x is the *variable of integration*, i.e. we are integrating with respect to x.

We will now refer to the operation of finding an antiderivative as *integration*.

Example 3.

$$(a) \quad \int x^3 dx = \frac{1}{4}x^4 + C \qquad .$$

$$(b) \quad \int x^4 dx = \frac{1}{5}x^5 + C \qquad .$$

Rules of Integration

$$\int x^n dx = \tfrac{1}{n+1}(x^{n+1}) + C \qquad \text{for } n \text{ not equal to -1} \qquad .$$

This is known as the *Power Rule* for integration.

The special case when $n = -1$ is described by

$$\int x^{-1} dx = \int \tfrac{1}{x} dx = \ln|x| + C \qquad .$$

The use of these two rules can be easily demonstrated.

Example 4. Solve the following integral

$$\int x^{17} dx$$

Solution: Since the power on x is not -1 we may apply the power rule with $n = 17$ to obtain

$$\int x^1 7 dx = \frac{1}{18} x^1 8 + C$$

Example 5. Solve the following integral

$$\int \sqrt{x^3} dx$$

Solution: Once again we invoke the power rule, this time with n $= \frac{3}{2}$.

$$\begin{aligned}
\int \sqrt{x^3}dx &= \int x^{\frac{3}{2}}dx \\
&= \frac{1}{\frac{3}{2}+1}x^{\frac{3}{2}+1} \\
&= \frac{1}{\frac{5}{2}}x^{\frac{5}{2}} \\
&= \frac{2}{5}x^{\frac{5}{2}}
\end{aligned}$$

The following theorems will be of great importance for solving integrals.

Theorem 2.3. *The integral of a sum is the sum of the integrals:*

$$\int [\ f(x) + g(x)\ ]dx = \int f(x)dx + \int g(x)dx$$

Theorem 2.4. *If c is any scalar then*

$$\int c\ \ f(x)dx = c\ \ \int f(x)dx$$

Example 6. Evaluate the following integrals.

(a) $\int (7x^3 - 3x^2 + x)dx$

(b) $\int (3\sqrt{x} + 5x^{-1})dx$

(c) $\int 1dx$

(d) $\int x^{\pi}dx$

Solutions:

(a) *By applying theorem 2.3 we have:*

$$\begin{aligned}
\int(7x^3 - 3x^2 + x)dx &= \int(7x^3)dx + \int(-3x^2)dx + \int(x)dx \\
&= 7\int(x^3)dx + (-3)\int(x^2)dx + \int(x)dx \quad \textit{(by theorem 2.4)} \\
&= 7\tfrac{x^4}{4} + (-3)\tfrac{x^3}{3} + \tfrac{x^2}{2} + C \quad \textit{(by the power rule)} \\
&= 7\tfrac{x^4}{4} - x^3 + \tfrac{x^2}{2} + C
\end{aligned}$$

(b) *Similarily,*

$$\begin{aligned}
\int(3\sqrt{x} + 5x^{-1})dx &= \int 3\sqrt{x}dx + \int 5x^{-1}dx \\
&= 3\int \sqrt{x}dx + 5\int x^{-1}dx \\
&= 3\int x^{\frac{1}{2}}dx + 5\int \tfrac{1}{x}dx \\
&= 3(\tfrac{2}{3})x^{\frac{3}{2}} + 5\ln|x| + C \\
&= 2x^{\frac{3}{2}} + 5\ln|x| + C
\end{aligned}$$

(c) *And,*

$$\begin{aligned}
\int 1dx &= \int x^0dx, \quad \textit{since} \quad x^0 = 1 \\
&= \tfrac{1}{1}x^1 + C \\
&= x + C
\end{aligned}$$

(d) Finally, remembering that π is just a constant we have

$$\begin{aligned}\int x^\pi dx &= \tfrac{1}{\pi+1}x^{\pi+1} + C \\ &= \tfrac{x^{\pi+1}}{\pi+1} + C\end{aligned}$$

We now add another rule of integration. Since e^x has the wonderful property that it is equal to its own derivative, the following should be of no surprise.

$$\boxed{\int e^x dx = e^x + C}$$

Side Conditions

As we well know, the general integral of a function f(x) involves an arbitary constant C. Sometimes we are provided information within the question that allows us to solve for C, giving a unique solution. Such information is called a *side condition.*

Example 7. Find the antiderivative of $f(x) = 3x^2$, which passes through the point (1,4).

Solution: The fact that our solution must pass through the point (1,4) is our side condition here, we must find an antiderivative of $f(x) = 3x^2$, say $F(x)$ which has $F(1) = 4$:

$$F(x) = \int 3x^2 dx = x^3 + C$$

Using our side condition,

$$F(1) = 4 \Rightarrow (1)^2 + C = 4 \quad \Rightarrow C = 3$$

Therefore,

$$F(x) = x^3 + 3$$

The method of Substitution

We now know how to integrate some forms of functions. It is important to note that in any of the formulae provided above that x is a "dummy variable". What we mean is that x could be replaced by any other variable and the formulae would remain true. This being said, we have the following list of " friendly forms " that we know how to integrate at a glance so far, we will add to this list soon.

Rule 1: $\int u^n du = \frac{1}{n+1}(u^{n+1}) + C, \quad n \neq -1$

Rule 2: $\int \frac{1}{u} du = \ln |u| + C$

Rule 3: $\int e^u du = e^u + C$

In many cases we cannot solve an integral as it is written, i.e. it may not be in a friendly form. However, in many cases we can rewrite it in a different way, so that it is changed into a friendly form that we can integrate. This is called *the method of substitution*.

We rewrite the integral by introducing a new variable, which is a function of the variable of integration. Then using the relationship between the current variable of integration and this new one, we attempt to replace all occurrences of the current variable with expressions involving the new one.

Thus we substitute a new variable into the integral, replacing the old one. There are many different such substitutions one can make. Often when trying to do an integral it is necessary to try a few different substitutions before one is found that makes the integral simple enough to do.

Definition 2.5. The substitution rule is the analogue of the chain rule in differentiation. Observe that if u is some function of x, say $u = g(x)$, then upon differentiating with respect to x we see that $\frac{du}{dx} = g'(x)$, so $du = g'(x)dx$. With this in mind we now state the *substitution rule*:

If $u = g(x)$ then $du = g'(x)dx$ and so

$$\int f(g(x))g'(x)dx = \int f(u)du$$

The substitution rule is quite unfriendly at first glance, but after one has completed a plethora of examples, the application of it becomes quite easy. We will provide a suggested algorithmic method of implementation as well as a "rule of thumb" list of suggested substitutions. The following process will read much more easily alongside an example, several follow.

Suggested Process for Integration:

Assume here that the variable of integration is x

(a) Pick u (see rules of thumb below).

(b) Differentiate u with respect to x (find $\frac{du}{dx}$) and solve for dx

(c) Substitute u into the original question, and replace dx with what you found it to be in (b).

(d) Simplify and make all cancellations possible. If u is the only variable involved in the newly expressed question, you are more or less guaranteed to have made the correct substitution and you should be able to solve the integral now. If however there remain x terms in the question, you probably (see exceptional cases below) did not make the correct substitution and should go back to (a).

(e) Integrate with respect to u.

(f) Since the original question was posed in terms of x, it seems appropriate that the answer should also be in terms of x. Thus, the last step is to replace u with what we decided it would be in (a).

Rules of thumb For Choices of u in Substitution

General form of Question	Usual choice for u
$\int \ f(x) \ \ln(\ g(x) \ ) \ dx$	Let u = g(x)
$\int \ f(x) \ e^{g(x)} \ dx$	Let u = g(x)
$\int \ \frac{f(x)}{g(x)} \ dx$	Let u = g(x)
$\int \ [g(x)]^n \ f(x) \ dx$	Let u = g(x)

Example 8. Solve: $\int (x^5 - 3)^4 x^4 dx$

Solution: Looking at this question we are sure that it is not a friendly form, and so use substitution.

(a) By the fourth rule of thumb it looks like we should choose $\boxed{u = (x^5 - 3)}$.

(b) Differentiating we get $\frac{du}{dx} = 5x^4$, and so $\boxed{dx = \frac{du}{5x^4}}$

(c) Substituting we get:

$$\begin{aligned}
\int (x^5 - 3)^4 x^4 dx &= \int (u)^4 x^4 \tfrac{du}{5x^4} \\
&= \int u^4 \tfrac{du}{5} && \text{(all x terms have cancelled)} \\
&= \tfrac{1}{5} \int u^4 du && \text{(friendly form so integrate)} \\
&= (\tfrac{1}{5})(\tfrac{1}{5})u^5 + C && \text{(power rule)} \\
&= \tfrac{u^5}{25} + C \\
&= \tfrac{(x^5-3)^5}{25} + C && \text{(step (f))}
\end{aligned}$$

Example 9. Solve: $\int \frac{3x^2+2x}{2x^3+2x^2+7}dx$

Solution: Again, looking at this question we see that it is not a friendly form, and so we use substitution.

(a) By the third rule of thumb it looks like we should choose $\boxed{\text{u} = (2x^3 + 2x^2 + 7)}$.

(b) Differentiating we get $\frac{du}{dx} = 6x^2 + 4x$, and so $\boxed{dx = \frac{du}{6x^2+4x}}$

(c) Substituting we get:

$$\begin{aligned}
\int \frac{3x^2+2x}{2x^3+2x^2+7}dx &= \int \frac{3x^2+2x}{u}\frac{du}{6x^2+4x} \\
&= \int \frac{3x^2+2x}{u}\frac{du}{2(3x^2+2x)} && \text{(all x terms have cancelled)} \\
&= \int \frac{1}{u}\frac{1}{2}du && \text{(all x terms have cancelled)} \\
&= \frac{1}{2}\int \frac{1}{u}du && \text{(friendly form so integrate)} \\
&= \frac{1}{2}\ln|u| + C \\
&= \frac{1}{2}\ln|2x^3 + 2x^2 + 7| + C && \text{(step (f))}
\end{aligned}$$

Example 10. Solve: $\int e^{x^3}x^2dx$

Solution: Again, looking at this question we see that it is not a friendly form, and so we use substitution.

(a) By the second rule of thumb it looks like we should choose $\boxed{\text{u} = (x^3)}$.

(b) Differentiating we get $\frac{du}{dx} = 3x^2$, and so $\boxed{dx = \frac{du}{3x^2}}$

(c) Substituting we get:

$$\begin{aligned}
\int e^{x^3} x^2 dx &= \int \int e^u x^2 \tfrac{du}{3x^2} \\
&= \int \int e^u \tfrac{du}{3} && \text{(all x terms have cancelled)} \\
&= \tfrac{1}{3} \int e^u du && \text{(friendly form so integrate)} \\
&= e^u + C \\
&= e^{x^3} + C && \text{(step (f))}
\end{aligned}$$

Sometimes it is not so obvious how to procede with a problem, it may appear that we have made an incorrect substitution when we have not. We give here some of these exceptional examples:

Example 11. Solve: $\int (x+2)^{50} x dx$

Solution: We see that this is not a friendly form, and so we use substitution.

(a) By the fourth rule of thumb it looks like we should choose $\boxed{\text{u} = (x+2)}$.

(b) Differentiating we get $\frac{du}{dx} = 1$, and so $\boxed{dx = du}$

(c) Substituting we get:

$$\begin{aligned}
\int (x+2)^{50} x dx &= \int (u)^{50} x du && \text{(it appears we may not cancel x)} \\
&= \int (u)^{50} (u-2) du && \text{(noting that } u = (x+2) \to x = u - 2\text{)} \\
&= \int ((u)^{51} - 2u^{50}) du && \text{(all x terms have cancelled)} \\
&= \int (u)^{51} du - 2 \int u^{50} du && \text{(friendly form so integrate)} \\
&= \tfrac{(u)^{52}}{52} - 2\tfrac{(u)^{51}}{51} + C \\
&= \tfrac{1}{52}(x+2)^{52} - \tfrac{2}{51}(x+2)^{51} + C && \text{(step (f))}
\end{aligned}$$

Example 12. Solve: $\int \frac{2x^4-3x^2+5}{5x}dx$

Solution: Once again if we use our rule of thumb we get nowhere. What we must recognize here is that we are asked to integrate an improper fraction. That is to say a fraction where the highest power of x in the numerator is *not* smaller than the highest power of x in the denominator. In such a case we must first reduce the fraction by actually dividing the bottom into the top. In this question this task is quite easy, others may require long division.

$$\begin{aligned}
\int \frac{2x^4-3x^2+5}{5x}dx &= \int (\frac{2x^4}{5x} - \frac{3x^2}{5x} + \frac{5}{5x})dx \\
&= \int (\frac{2}{5}x^3 - \frac{3}{5}x + \frac{1}{x})dx \\
&= \frac{2}{5}\int x^3 dx - \frac{3}{5}\int x dx + \int \frac{1}{x}dx \quad \text{(friendly form so integrate)} \\
&= \frac{2}{5}\frac{1}{4}x^4 - \frac{3}{5}\frac{1}{2}x^2 + \ln|x| + C \\
&= \frac{1}{10}x^4 - \frac{3}{10}x^2 + \ln|x| + C
\end{aligned}$$