Unit 5
Area Between Curves and Volumes of Revolution

We have learned how to find the area beneath a positive function. In this section we will learn how to find more general areas, those bounded between two curves. For example, suppose we wish to find the area bounded by $f(x)$ and $g(x)$ between $x = 1$ and $x = 3$ as shown:

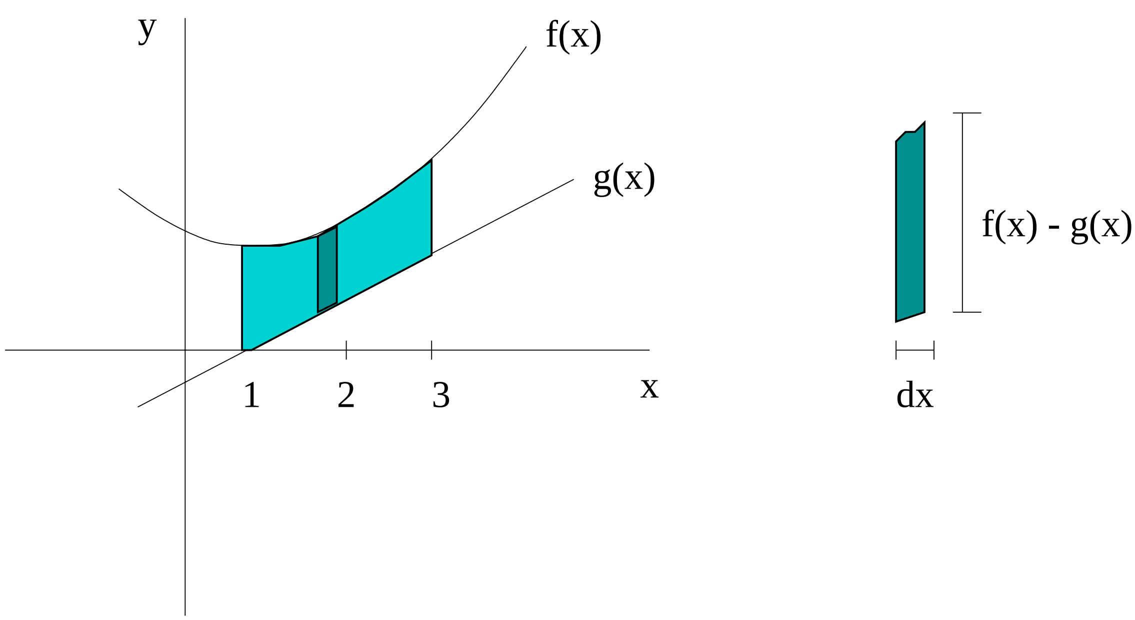

Figure 1: Area bounded by two curves

On the interval [1,3], we see that $f(x) \geq g(x)$, so the function $(f(x) - g(x))$ is positive on [1,3]. With this oservation in mind we can see that the problem of finding the area between two curves ($f(x)$ and $g(x)$ here) can be reduced to one of finding the area under a positive function ($(f(x) - g(x))$ here). So we need not learn another technique for these problems, we simply reduce the problem as above and then apply our earlier method of sketching the graph, drawing a small strip and calculating its area, and then integrating to find the entire area.

Example 1. Find the area bounded by the curves

$$f(x) = x^2 + 1 \quad and \quad g(x) = x - 1$$

on the interval [1,3].

Solution: We first sketch the region involved, this yields figure 1 (previous page). Drawing the small strip we see that it has area:

$$dA = (f(x) - g(x))dx = [(x^2 + 1) - (x - 1)]dx$$

We may therefore represent the area in question by:

$$\begin{aligned} A &= \int_1^3 [(x^2 - 1) - (x - 1)]dx \\ &= \int_1^3 [(x^2 - x + 2)]dx \\ &= \left[\frac{x^3}{3} - \frac{x^2}{2} + 2x\right]_1^3 \\ &= [(9 - \tfrac{9}{2} + 6) - (\tfrac{1}{3} - \tfrac{1}{2} + 2)] \\ &= 6 + \tfrac{9}{2} - \tfrac{11}{6} \\ &= \tfrac{26}{3} \end{aligned}$$

Example 2. Find the area bounded by the curves

$$f(x) = 9 - x^2 \quad and \quad y = 5$$

Solution: We first sketch the region involved, this will tell us the interval involved (by the intersection points of the two functions. Setting

$$f(x) = 9 - x^2 = y = 5$$

we get

$$x = \pm 2$$

so we have the intersection points shown:

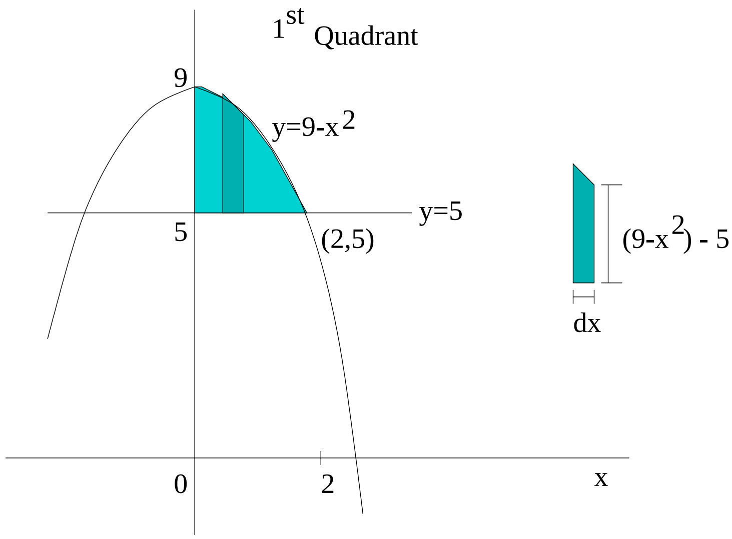

Figure 2: Area bounded by $f(x) = 9 - x^2$, and $y = 5$

Drawing the small strip as shown we see that it has area:

$$dA = [(9 - x^2) - (5)]dx = (4 - x^2)dx$$

We may therefore represent the area in question by:

$$\begin{aligned} A &= \int_{-2}^{2} (4 - x^2)dx \\ &= \left[4x - \tfrac{x^3}{3}\right]_{-2}^{2} \\ &= [(8 - \tfrac{8}{3}) - (-8 + \tfrac{8}{3})] \\ &= 2(8) - \tfrac{2}{3}(8) \\ &= \tfrac{32}{3} \end{aligned}$$

Example 3. Find the area bounded by the curves

$$f(x) = x^2 - 4 \quad and \quad g(x) = -x + 2$$

Solution: Once again we sketch the region involved and find the intersection points of the two functions in order to find the interval. Setting

$$x^2 - 4 = -x + 2$$

we get

$$x^2 + x - 6 = 0 \Rightarrow (x+3)(x-2) = 0 \Rightarrow x = -3, x = 2$$

so we have the intersection points $(-3, 5)$, and $(2, 0)$ as shown:

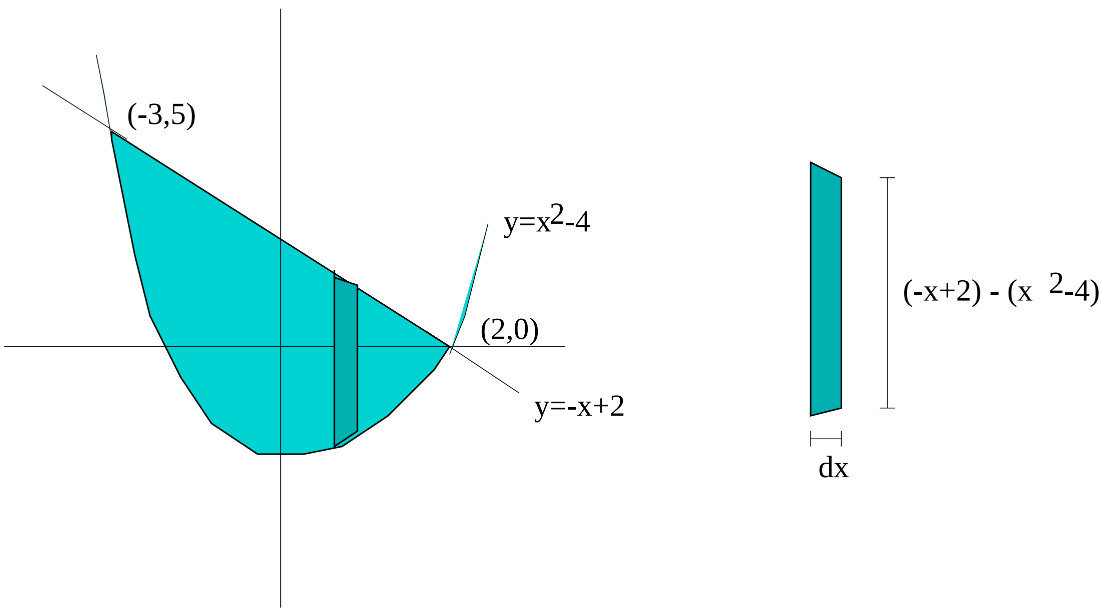

Figure 3: Area bounded by $f(x) = x^2 - 4$, and $g(x) = -x + 2$

Drawing the small strip as shown we see that it has area:
$dA = [(-x+2) - (x^2 - 4)]dx = (-x^2 - x + 6)dx$
We may therefore represent the area in question by:

$$\begin{aligned} A &= \int_{-3}^{2} (-x^2 - x + 6)dx \\ &= \left[-\frac{x^3}{3} - \frac{x^2}{2} + 6x \right]_{-3}^{2} \\ &= [(-\tfrac{8}{3} - 2 + 12) - (9 - \tfrac{9}{2} - 18)] \\ &= \tfrac{125}{6} \end{aligned}$$

Integrating With Respect to y : Horizontal Slicing

If a regions bounding curves are described by giving x as a function of y then the process above will change slightly. The only difference being that we will integrate with respect to y instead of x. This method can save some steps sometimes as the following example shows:

Example 4. Find the area in the first quadrant bounded by the curves $y = \sqrt{x}$, and $y = x - 2$.
Solution: We first sketch the region involved, this will tell us the interval involved (by the intersection points of the two functions and which parts are in the first quadrant) we have the following:.

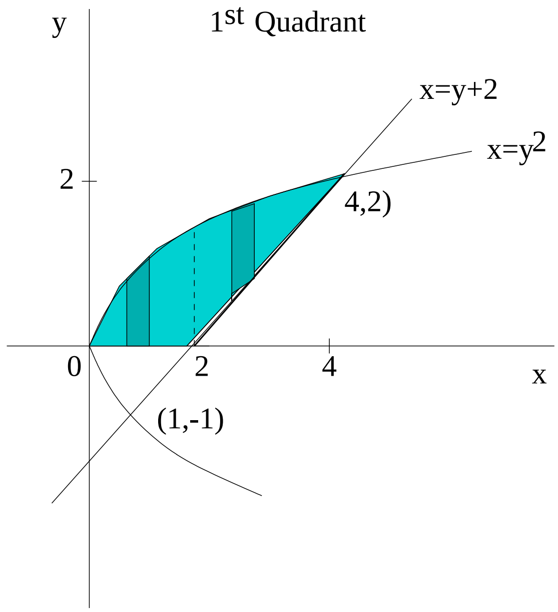

Figure 4: Area bounded by $y = \sqrt{x}$, and $y = x - 2$

Figure 4 shows that the upper boundary of the area in question is given by the curve $y = \sqrt{x}$, whereas the bottom boundary is given by two curves, namely $y = 0$ for $0 \leq x \leq 2$, and $y = x - 2$ for $2 \leq x \leq 4$. So we will need two integrals to solve this area, one for $0 \leq x \leq 2$ and one for $2 \leq x \leq 4$. In the first case the small strip has area:

$$dA = (\sqrt{x} - 0)dx = (\sqrt{x})dx$$

And in the second case it has area:

$$dA = (\sqrt{x} - (x - 2))dx = (\sqrt{x} - x + 2)dx$$

We may therefore represent the area in question by:

$$
\begin{aligned}
A &= \int_0^2 (\sqrt{x})dx + \int_2^4 (\sqrt{x} - x + 2)dx \\
&= \left[\tfrac{2}{3}x^{\frac{3}{2}}\right]_0^2 + \left[\tfrac{2}{3}x^{\frac{3}{2}} - \tfrac{x^2}{2} + 2x\right]_2^4 \\
&= [\tfrac{2}{3}(2^{\frac{3}{2}}) - 0] + [(\tfrac{2}{3}(4^{\frac{3}{2}}) - 8 + 8) - (\tfrac{2}{3}(2^{\frac{3}{2}}) - 2 + 4)] \\
&= \tfrac{2}{3}(8) - 2 \\
&= \tfrac{10}{3}
\end{aligned}
$$

If we solved this problem a little differently, we may make it a little simpler. Expressing our boundary curves in terms of y instead of x gives a slight advantage.

Example 5. Again find the area in the first quadrant bounded by the curves $x = y^2$, and $x = y + 2$.
Solution: We first sketch the region involved:

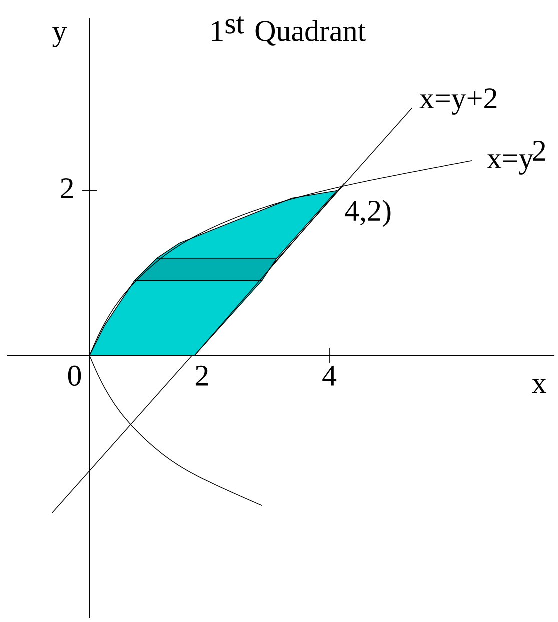

Figure 5: Area bounded by $x = y^2$, and $x = y + 2$

Figure 5 shows that the leftmost boundary of the area in question is given by the curve $x = y^2$,and the rightmost boundary is given by $x = y + 2$. Drawing the small strip we see that it has area:
$dA = [(y + 2) - (y^2)]dy = (-y^2 + y + 2)dy$

As we range the horizonlal strip through the region in question, we see that y varies from 0 to 2, we may therefore represent the area in question by:

$$\begin{aligned} A &= \int_0^2 (-y^2 + y + 2)dy \\ &= \left[-\frac{y^3}{3} + \frac{y^2}{2} + 2y \right]_0^2 \\ &= (2 + 4 - \frac{8}{3} - (0) \\ &= \frac{10}{3} \end{aligned}$$

This was clearly much more simple than the previous solution.

We now move on to yet another application of definite integrals: **volumes of revolution**. Volumes of revolution are solids whose shapes can be generated by revolving some curve(s) about some axis in three-space. If we can set things up so that a solid of revolution is generated by revolving the region between the graph of a continuous function $f(x), a \leq x \leq b$ and the x axis, and the axis of rotation is the x axis (see diagram below), we can then calculate the volume in the following way:

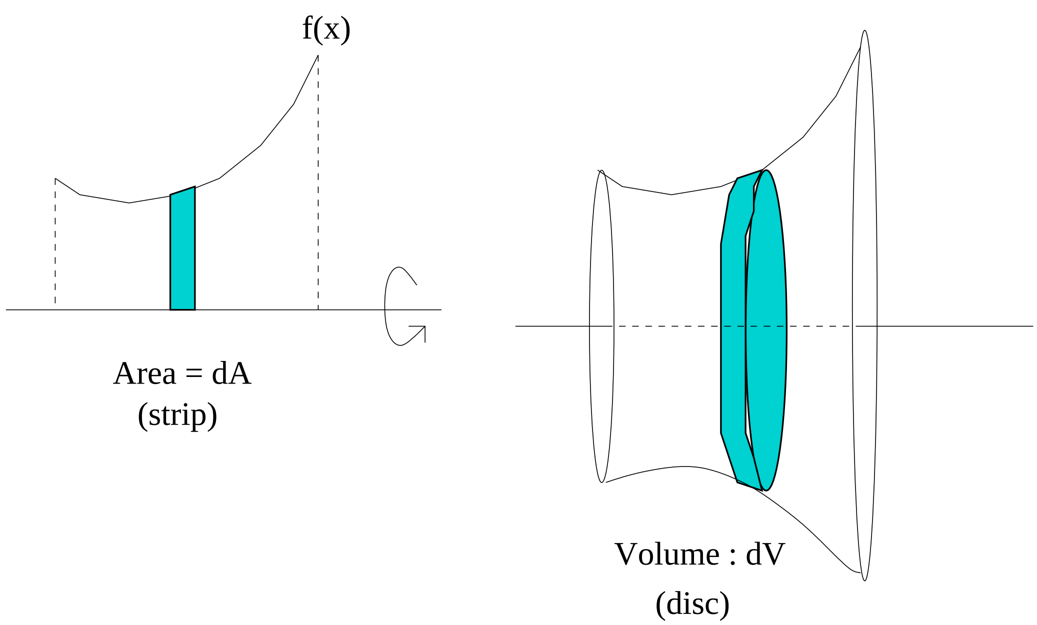

Figure 6:

The steps to follow are very familliar: (1) sketch the region to be revolved
(2) Draw a small strip perpendicular to the axis of revolution ,then revolve it about the axis of rotation and calculate the volume that it generates, say dV (see Fig 6)
(3) Integrate dV to find the entire volume

Example 6. Find the volume generated by revolving the region bounded by $y = \sqrt{x}, y = 0$, and $x = 4$ about the x-axis.
Solution: We first sketch the region in question, and draw our small strip perpendicular to the x-axis (with width dx):

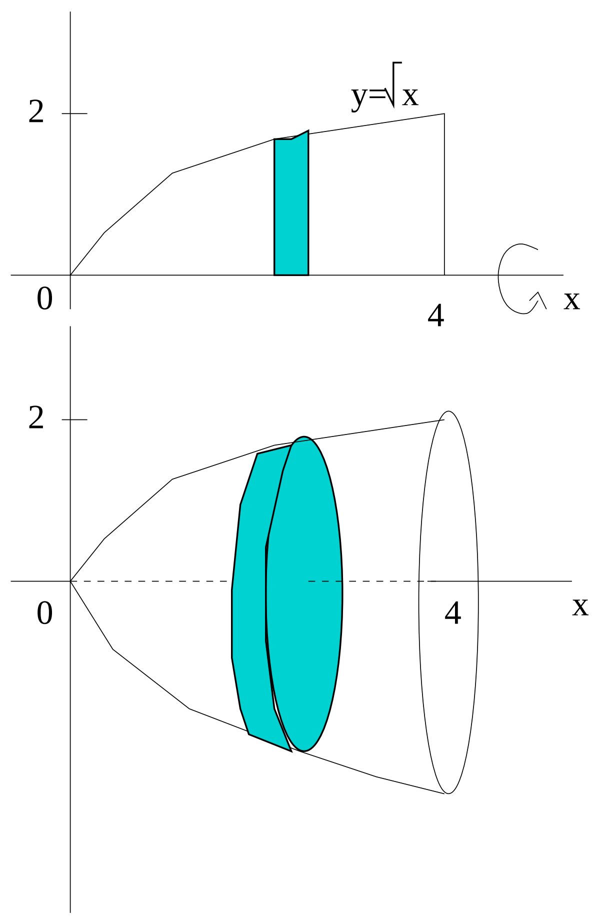

Figure 7:

Rotating the strip about the x-axis we see that we get something of the form:

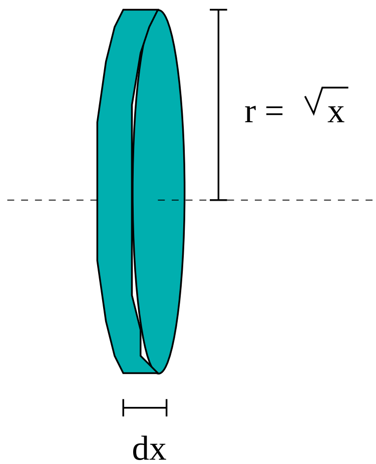

Figure 8: $volume = \pi r^2 h$

This is clearly a cylindrical shape and so has volume given by the classical formula: $v = \pi r^2 h$, where r is the radius of the cylinder, and h is the height. Looking at the specific solid generated by the strip here, we see that $h = dx$ and r=the length of the strip =the y-value of the curve $=\sqrt{x}$. So the volume generated by the strip is given by:

$$\begin{aligned} dV &= \pi r^2 h \\ &= \pi(\sqrt{x})^2 dx \\ &= \pi x dx \end{aligned}$$

We also see from the sketch that x varies from 0 to 4 in the region, so these are our limits of integration. Our volume is therefore represented by:

$$\begin{aligned} V &= \int_0^4 \pi x dx \\ &= \pi \left[\frac{x^2}{2}\right]_0^4 \\ &= \pi(8 - 0) \\ &= 8\pi \end{aligned}$$

The next examples illustrate the above process which is sometimes called the **method of washers**, for a soon obvious reason (the strip generates a solid resembling a washer).

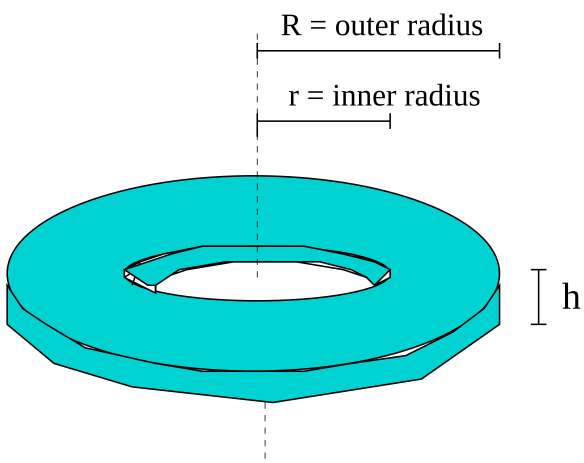

Figure 9: washer volume $= \pi(R^2 - r^2)h$

To find the volume dV, of such an animal, we simply find the volume of the large disc as if it were solid $(\pi R^2 h)$and then subtract the volume of the hole $(\pi r^2 h)$. This gives us the formula:

$$\boxed{dV = \pi(R^2 - r^2)h}$$

The use of the above formula is better illustrated through some examples:

Example 7. Find the volume generated by revolving the region bounded by $y = x^2 + 2$, $y = 1$, $x = 0$ and $x = 2$ about the x-axis.

Solution: We first sketch the region in question, and draw our small strip perpendicular to the x-axis (with width dx):

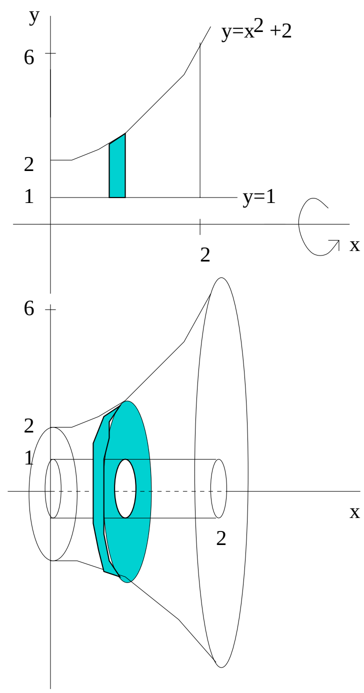

Figure 10:

Rotating the strip about the x-axis we see that we get something resembling figure 11.

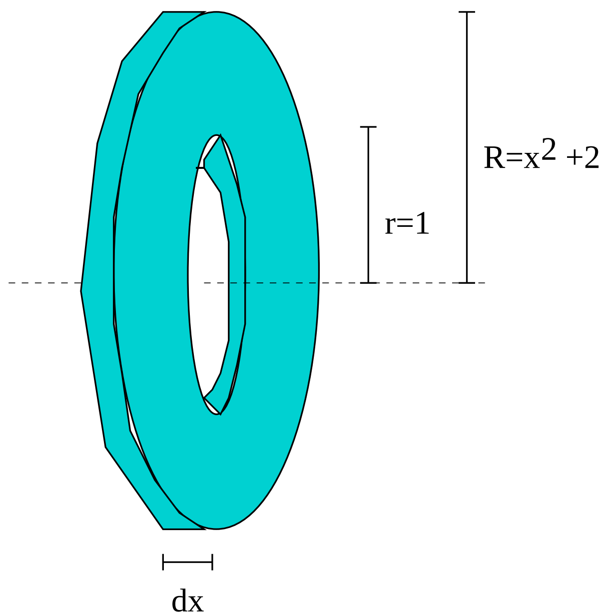

Figure 11:

The volume generated by the strip is one of a washer with $R =$ (the distance from the x-axis to the outer edge of the strip) $= x^2 + 2$, $r =$ (the distance from the x-axis to the inner edge of the strip) $= 1$, and $h = dx$. So the volume generated by the strip is given by:

$$\begin{aligned} dV &= \pi(R^2 - r^2)h \\ &= \pi[(x^2+2)^2 - (1^2)]dx \\ &= \pi(x^4 + 4x^2 + 3)dx \end{aligned}$$

We also see from the sketch that x varies from 0 to 2 in the region, so these are our limits of integration. Our volume is therefore represented by:

$$\begin{aligned} V &= \int_0^2 \pi[(x^2+2)^2 - (1^2)]dx \\ &= \pi \int_0^2 (x^4 + 4x^2 + 3)dx \\ &= \pi \left[\tfrac{1}{5}x^5 + \tfrac{4}{3}x^3 + 3x\right]_0^2 \\ &= \pi \left[(\tfrac{32}{5} + \tfrac{32}{3} + 6) - 0\right] \\ &= \pi \left[(\tfrac{32}{5} + \tfrac{32}{3} + 6)\right] \approx 72.5 \end{aligned}$$

Just as with areas, we sometimes use horizontal strips for finding volumes. This comes about since the method we learned above requires the strips to be perpendicular to the axis of rotation, so if we revolve a region about, say, the y-axis then our strips must be horizontal. All other mechanics of such a problem are business as usual as we shall see:

Example 8. Find the volume generated by revolving the region bounded by $y = x^3$, $y = 8$,and $x = 0$ about the line $x = -1$.
Solution: We first sketch the region in question, and draw our small strip (with width dy) perpendicular to the axis of rotation :

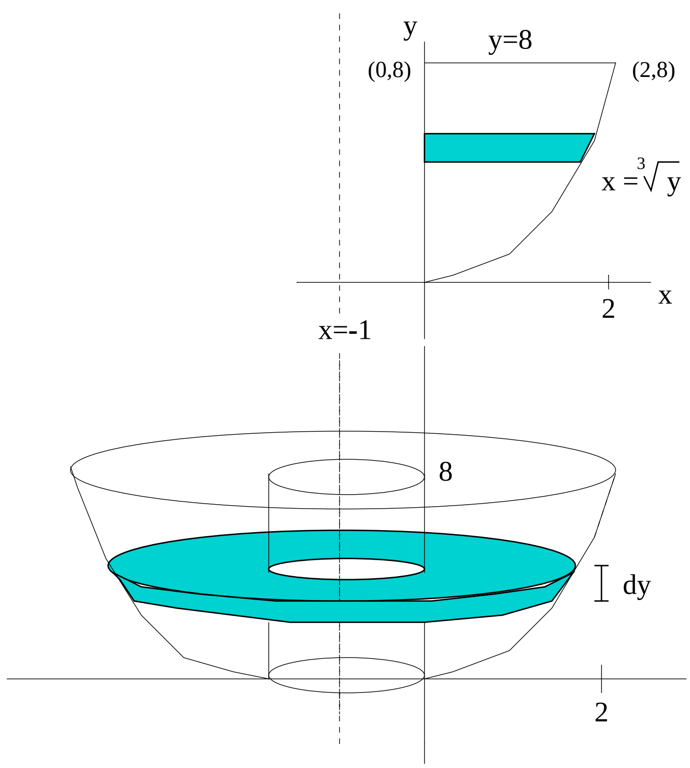

Figure 12:

Rotating the strip about the axis of rotation we see that we get something resembling figure 13.

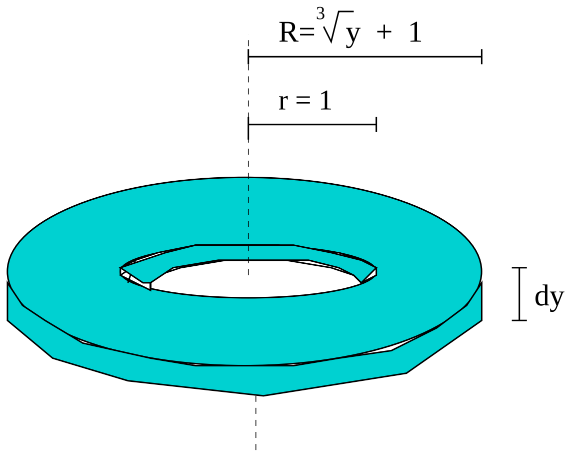

Figure 13:

The volume generated by the strip is one of a washer with $R =$ (the distance from the axis of rotation to the outer edge of the strip)=(1+the x value of the outer curve) $= 1+y^{\frac{1}{3}}$, $r =$ (the distance from the axis of rotation to the inner edge of the strip) $= 1$, and $h = dy$. So the volume generated by the strip is given by:

$$\begin{aligned} dV &= \pi(R^2 - r^2)h \\ &= \pi[(1 + y^{\frac{1}{3}})^2 - (1^2)]dy \\ &= \pi(y^{\frac{2}{3}} + 2y^{\frac{1}{3}})dy \end{aligned}$$

We also see from the sketch that y varies from 0 to 8 in the region, so these are our limits of integration. Our volume is therefore represented by:

$$\begin{aligned} V &= \int_0^8 \pi[(y^{\frac{2}{3}} + 2y^{\frac{1}{3}})dy \\ &= \pi \int_0^8 (y^{\frac{2}{3}} + 2y^{\frac{1}{3}})dy \\ &= \pi \left[\tfrac{3}{5}y^{\frac{5}{3}} + \tfrac{3}{2}y^{\frac{4}{3}}\right]_0^8 \\ &= \pi \left[\tfrac{3}{5}(8)^{\frac{5}{3}} + \tfrac{3}{2}(8)^{\frac{4}{3}}\right] \\ &= \pi \left[(\tfrac{96}{5} + 24\right] \approx 135.7 \end{aligned}$$