Unit 1:
Exponential Functions and Logarithmic Functions

Definition 1.1. By an *exponential function* we mean a function of the form

$$\boxed{f(x) = b^x} \tag{1}$$

where b is a positive constant called the *base* of the function.

Example 1. $f(x) = 2^x$

Theorem 1.2. *The following are properties of exponents:*

Properties of Exponents:

$$\begin{array}{rrcl} (i) & b^x b^y & = & b^{x+y} \\ (ii) & b^x/b^y & = & b^{x-y} \\ (iii) & (b^x)^y & = & b^{xy} \end{array}$$

Example 2. Simplify the following

(a) $\left(\frac{3^5 3^7}{3^3}\right)^2$ (b) $\left(\frac{7^6 7^{-4}}{7^2}\right)^3$

Solution:

(a)
$$\begin{aligned}\left(\frac{3^5 3^7}{3^3}\right)^2 &= \left(\frac{3^{5+7}}{3^3}\right)^2\\ &= \left(3^{(5+7)-3}\right)^2\\ &= 3^{(5+7-3)\times 2}\\ &= 3^{18}\end{aligned}$$

(b)
$$\begin{aligned}\left(\frac{7^6 7^{-4}}{7^2}\right)^3 &= \left(\frac{7^{6+(-4)}}{7^2}\right)^3\\ &= \left(7^{(6+(-4))-2}\right)^3\\ &= 7^{(6+(-4)-2)\times 3}\\ &= 7^0\\ &= 1\end{aligned}$$

Exponentials are closely related to logarithmic functions.

Definition 1.3. Let $b > 0$, $b \neq 1$, we define $log_b x$ as:
the exponent to which b must be raised, to yield x; that is

$$b^{\log_b x} = x \tag{2}$$

or

$$y = \log_b x \iff b^y = x \tag{3}$$

Example 3. Evaluate the following logarithmic expressions.

(a) $\log_2(16) = 4$ (b) $\log_2(\frac{1}{8}) = -3$

(c) $\log_3(9) = 2$ (d) $\log_b(1) = 0$

(e) $\log_4(8) = \frac{3}{2}$ since $8 = 2^3 = (\sqrt{4})^3 = 4^{\frac{3}{2}}$

Theorem 1.4. *The following are properties of logarithms:*

Properties of Logarithms:

(i)	$\log_b(uv)$	$=$	$\log_b(u) + \log_b(v)$
(ii)	$\log_b(u/v)$	$=$	$\log_b(u) - \log_b(v)$
(iii)	$\log_b(u^r)$	$=$	$r\ \log_b(u)$

Example 4. Express the following as a single logarithm:
(a) $\log_b(x) + \log_b(y) - \log_b(z)$
(b) $3\log_b(x) - \frac{1}{2}\log_b(y)$

Solution:

(a) $\log_b(x) + \log_b(y) - \log_b(z) = \log_b(xy) - \log_b(z) = \log_b(\frac{xy}{z})$

(b) $$\begin{aligned} 3\log_b(x) - \tfrac{1}{2}\log_b(y) &= \log_b(x^3) - \log_b(y^{1/2}) \\ &= \log_b\left(\frac{x^3}{y^{1/2}}\right) \end{aligned}$$

Natural Exponential Functions and Natural Logarithmic Functions

Definition 1.5. We can discuss logarithms with arbitrary base, b. However, a special number

$$e = \lim_{x \to 0} (1+x)^{1/x} \approx 2.71828... \tag{4}$$

is used so frequently that we define

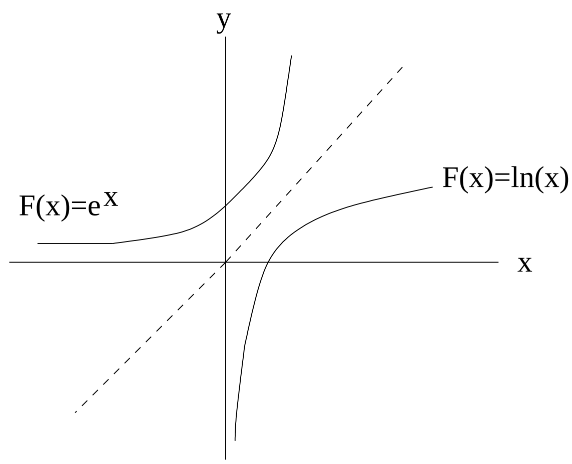
e^x : The *Natural Exponential Function*

$\log_e(x)$: The *Natural Logarithmic Function*

Notation: $\log_e(x)$ will be denoted as $\ln(x)$.

Fact: $\ln(x)$ and e^x are inverses of each other.

i.e. $\quad e^{\ln(x)} = x, \quad \ln(e^x) = x$



Example 5. Solve $3^{2x+1} = 4.9$ for x

Solution: We have to knock the $(2x+1)$ down in order to get at it. For this we utilize the property :

$$\ln(a^b) = b\ln(a)$$

Taking the natural log, i.e., ln, of both sides of the original equation:

$$\begin{aligned} & \ln\left(3^{2x+1}\right) &= \ln(4.9) \\ \Rightarrow & (2x+1)\ln(3) &= \ln(4.9) \\ \Rightarrow & (2x+1) &= \tfrac{\ln(4.9)}{\ln(3)} \approx 1.4466 \\ \Rightarrow & 2x &= 0.4466 \\ \Rightarrow & x &= 0.2233 \end{aligned}$$

Theorem 1.6. *We may take the derivative of both logarithmic and exponential functions according to the following rules:*

(i)	$\frac{d}{dx}(\ln(x)) = \frac{1}{x}$
(ii)	$\frac{d}{dx}(e^x) = e^x$

Fact: The only functions which have the property that $f'(x) = f(x)$ are those of the form $f(x) = c\ e^x$ where c is some constant.

Example 6. Find $f'(x)$ where

(a) $f(x) = e^x \ln(x)$
(b) $f(x) = x^2 e^{-x}$

Solution:

(a) Using the Product Rule:

$$\begin{aligned} f'(x) &= e^x \frac{d}{dx}(\ln(x)) + (\ln(x)) \frac{d}{dx}(e^x) \\ &= e^x \left(\frac{1}{x}\right) + (\ln(x))\,(e^x) \\ &= e^x \left(\frac{1}{x} + \ln(x)\right) \end{aligned}$$

(b) $f(x) = x^2 e^{-x} = \frac{x^2}{e^x}$.

Using the Quotient Rule:

$$\begin{aligned} f'(x) &= \frac{e^x \frac{d}{dx}(x^2) - x^2 \frac{d}{dx}(e^x)}{e^{2x}} \\ &= \frac{2xe^x - x^2 e^x}{e^{2x}} \\ &= \frac{x(2-x)}{e^x} \end{aligned}$$

Caveat: Though quite similar in appearance observe the difference between the following three derivatives.

(1) $\frac{d}{dx}\left(x^e\right) = ex^{e-1}$

(2) $\frac{d}{dx}\left(e^x\right) = e^x$

(3) $\frac{d}{dx}\left(e^e\right) = 0$

Chain Rule

So far we know how to take derivatives of functions of the form:

(i)	x^n	*Power Rule*
(ii)	$f(x) \cdot g(x)$	*Product Rule*
(iii)	$f(x)/g(x)$	*Quotient Rule*
(iv)	$\ln(x)$	*Theorem 1.6*
(v)	e^x	*Theorem 1.6*

Note: x is just a *dummy variable*, what this means is that we could write

$$\tfrac{d}{dx}e^x = e^x \qquad \text{or} \qquad \tfrac{d}{du}e^u = e^u \qquad \text{or} \qquad \tfrac{d}{dz}e^z = e^z$$

and the meaning of the statement remains unchanged.

So how do we take derivatives of functions like

$$f(x) = e^{x^3+2x} \qquad \text{or} \qquad g(x) = \ln\left(\tfrac{x^2+3}{2x-1}\right) \ ?$$

Neither of these functions fit a form from the list above. We need a new method, namely the *Chain Rule*.

Definition 1.7. Suppose y is a function of u, and u in turn is a function of x, *i.e.*

$$y = f(u) \qquad \text{and} \qquad u = g(x)$$

So that in effect, y is a function of x. The *Chain Rule* allows us to calculate dy/dx as follows

$$\textit{Suppose} \quad y = f(u) \quad \textit{and} \quad u = g(x) \text{ then}$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \tag{5}$$

Theorem 1.8. *Explicitly stated we have*

> ***Chain Rule:*** *Let $y = f(u)$ and $u = g(x)$ then we have the composite function $y = f(g(x))$. Then if both dy/du and du/dx exist, we have*
> $$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

You *must* do as many exercises in this as necessary to *master* the chain rule.

Example 7. Given $y = e^{x^2+2x}$. Find dy/dx.

Solution: We know that $\frac{d}{du}(e^u) = e^u$. Therefore, we must try to put y into this form. Let $u = x^2 + 2x$. Then,

$$y = e^u \qquad \text{with} \qquad \frac{du}{dx} = 2x + 2 \quad \text{and} \quad \frac{dy}{du} = \frac{d}{du}(e^u) = e^u$$

The Chain Rule now yields:

$$\begin{aligned}\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \quad &= (e^u)\,(2x+2) \\ &= e^{x^2+2x}(2x+2)\end{aligned}$$

Example 8. Given $y = \ln\left(\frac{x^2+3}{2x-1}\right)$. Find dy/dx.

Solution: We know that $\frac{d}{du}(\ln(u)) = \frac{1}{u}$. Therefore, we must try to put y into this form. Let $u = \frac{x^2+3}{2x-1}$. Therefore,

$$y = \ln(u) \qquad \text{where} \qquad \frac{du}{dx} = \frac{(2x-1)(2x)-(x^2+3)(2)}{(2x-1)^2} \qquad \text{and} \qquad \frac{dy}{du} = \frac{1}{u}$$

The Chain Rule now yields:

$$\begin{aligned}\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \quad &= \left(\frac{1}{u}\right)\frac{(2x-1)(2x)-(x^2+3)(2)}{(2x-1)^2} \\ &= \frac{2x-1}{x^2+3}\,\frac{(2x-1)(2x)-(x^2+3)(2)}{(2x-1)^2}\end{aligned}$$

Theorem 1.9. ***General Applications of Chain Rule:*** *If u is a function of x then*

$$\text{(1)} \quad \tfrac{d}{dx}\left(u^n\right) = n(u)^{n-1} \cdot \tfrac{du}{dx}$$

$$\text{(2)} \quad \tfrac{d}{dx}\left(e^u\right) = e^u \cdot \tfrac{du}{dx}$$

$$\text{(3)} \quad \tfrac{d}{dx}\left(\ln(u)\right) = \left(\tfrac{1}{u}\right) \cdot \tfrac{du}{dx}$$

Sometimes the substitution is not so obvious, as the next example illustrates.

Example 9. Find dy/dx given $y = (\ln(3x+1))^3$.

Solution: Due to the *unfriendly* form of the question, we see that we must use the chain rule again.

First try: let $u = 3x + 1$

then $y = (\ln(u))^3$

which is still not easy to differentiate. So we try a different substitution.

Second try: let $u = \ln(3x + 1)$

then $y = u^3, \quad \frac{dy}{du} = 3u^2 = 3\left(\ln(3x+1)\right)^2$

and $\frac{du}{dx} = \frac{d}{dx}(\ln(3x+1)) = \frac{1}{3x+1}\left(\frac{d}{dx}(3x+1)\right) = \frac{3}{3x+1}$

So, we have

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = 9\frac{\left(\ln(3x+1)\right)^2}{3x+1}$$

General Logarithm and Exponential Functions

Theorem 1.10. *We have the following conversion formula for logs with an arbitrary base, b to natural logs.*

$$\boxed{\log_b(x) = \frac{\ln(x)}{\ln(b)}} \tag{6}$$

Example 10. Express the following logs as natural logs:

$$\text{(a)} \quad \log_2(10) \qquad\qquad \text{(b)} \quad \log_3(7)$$

Solution:

$$\text{(a)} \quad \log_2(10) = \frac{\ln(10)}{\ln(2)} \qquad\qquad \text{(b)} \quad \log_3(7) = \frac{\ln(7)}{\ln(3)}$$

Theorem 1.11. *We have the following rules for differentiating:*

(1) $\frac{d}{dx}\left(\log_b(x)\right) = \frac{1}{x\ln(b)}$

(2) $\frac{d}{dx}\left(b^x\right) = b^x \ln(b)$

(3) $\frac{d}{dx}\left(\log_b(u)\right) = \frac{1}{u\ln(b)} \cdot \frac{du}{dx}$

(4) $\frac{d}{dx}\left(b^u\right) = b^u \ln(b) \cdot \frac{du}{dx}$

Example 11. Find dy/dx for each of the following

(a) $y = \log_7(x)$ (b) $y = \log_3(2x^3 - 3x)$
(c) $y = 2^x$ (d) $y = 7^{e^x - e^{-x}}$

Solution:

(a) $\frac{dy}{dx} = \frac{1}{x\ln(7)}$

(b) $\frac{dy}{dx} = \frac{1}{(2x^3-3x)\ln(3)} \cdot (6x - 3)$

(c) $\frac{dy}{dx} = 2^x \ln(2)$

(d) $\frac{dy}{dx} = 7^{(e^x - e^{-x})} \ln(7) \cdot (e^x + e^{-x})$

Logarithmic Differentiation:

Logarithmic Differentiation is usually used when a function is in an unfriendly form such as

Type 1: $y = (f(x))^{g(x)}$

or

Type 2: $y = (f(x))^n (g(x))^m (h(x))^k \cdots$ or $y = \frac{(f(x))^n (g(x))^m \cdots}{(h(x))^k \cdots}$

Definition 1.12. Logarithmic differentiation makes use of the Properties of Logarithmic Functions to transform unfriendly function into ones that may be differentiated more easily. Logarithmic differentiation can be broken down into five steps:

(1) Take ln of both sides

(2) Exploit a property of logs to re-express the equation

(3) Differentiate implicitly

(4) Solve for dy/dx

(5) Express the answer in terms of the original variable

Example 12. Find the derivative of $y = (x^2+3)^{2x-1}$.

Solution: Recognize that $y = (x^2+3)^{2x-1}$ is in the form of $y = (f(x))^{g(x)}$ We have no rules, so far, for functions of this form. So we turn to Logarithmic Differentiation.

$$y = (x^2+3)^{2x-1}$$

Take ln of both sides :

$$\ln(y) = \ln\left((x^2+3)^{2x-1}\right)$$

Use the power property of ln; $(\ln(a^b) = b\ln(a))$ to get:

$$\ln(y) = (2x-1)\ln\left(x^2+3\right)$$

Everything is now in a form we can differentiate.

$$\frac{1}{y}\frac{dy}{dx} = (2x-1)\cdot\frac{1}{x^2+3}\cdot(2x) + \ln(x^2+3)\cdot(2)$$

Solve for dy/dx :

$$\frac{dy}{dx} = y\left((2x-1)\cdot\frac{1}{x^2+3}\cdot(2x) + \ln(x^2+3)\cdot(2)\right)$$

Replace y with $y = (x^2+3)^{2x-1}$:

$$\frac{dy}{dx} = (x^2+3)^{2x-1}\left((2x-1)\cdot\frac{1}{x^2+3}\cdot(2x) + \ln(x^2+3)\cdot(2)\right)$$

Example 13. Find the derivative of $y = \frac{(2x-3)^4(3x-7)^5}{(7x-2)^6}$.

Solution: It is possible to differentiate this function using Quotient and Chain Rule but it would be terribly messy. Recognize that $y = \frac{(2x-3)^4(3x-7)^5}{(7x-2)^6}$ is in the form of $y = \frac{(f(x))^n(g(x))^m}{(h(x))^k}$ so use Logarithmic differentiation.

$$y = \frac{(2x-3)^4(3x-7)^5}{(7x-2)^6}$$

Take ln of both sides :

$$\ln(y) = \ln\left(\frac{(2x-3)^4(3x-7)^5}{(7x-2)^6}\right)$$

Use the addition and substraction properties of ln:

$$\ln(y) = \ln\left((2x-3)^4\right) + \ln\left((3x-7)^5\right) - \ln\left((7x-2)^6\right)$$

Use the power property of ln:

$$\ln(y) = 4\ln(2x-3) + 5\ln(3x-7) - 6\ln(7x-2)$$

Now differentiate (implicitly) :

$$\frac{1}{y}\frac{dy}{dx} = 4 \cdot \frac{1}{2x-3} \cdot (2) + 5 \cdot \frac{1}{3x-7} \cdot (3) - 6 \cdot \frac{1}{7x-2} \cdot (7)$$

Solve for dy/dx :

$$\frac{dy}{dx} = y\left(\frac{8}{2x-3} + \cdot \frac{15}{3x-7} - \frac{42}{7x-2}\right)$$

Replace y with $y = \frac{(2x-3)^4(3x-7)^5}{(7x-2)^6}$:

$$\frac{dy}{dx} = \frac{(2x-3)^4(3x-7)^5}{(7x-2)^6}\left(\frac{8}{2x-3} + \cdot \frac{15}{3x-7} - \frac{42}{7x-2}\right)$$