Unit 4
The Definite Integral

The formal definition of the definite integral

Suppose the function f is continuous on the interval [a,b]. We begin by partitioning the interval [a,b] into n subintervals of equal length by choosing points, say $x_0, x_1, ...x_n$ between a and b subject to the conditions that

$$a = x_0 < x_1 < x_2 < ... < x_{n-1} < x_n = b \tag{1}$$

and

$$\Delta x = x_i - x_{i-1} = (b - a)/n \tag{2}$$

We then construct a rectangle over each interval $[x_{i-1}, x_i]$ of "height" $f(x_i)$. The area of the i'th rectangle is therefore the positive or negative value (depending if $f(x_i)$ is positive or negative) of $f(x_i) \times \Delta x$. See figure 1:

As the partition of [a,b] becomes finer and finer (i.e. as n gets bigger and so our rectangle widths become smaller) we would expect the rectangles to approximate the region between the x-axis and the curve with increasing accuracy. This is indeed what happens and we define the definite integral of f(x) between a and b in this manner:

Definition 4.1. Let f(x) be a continuous function on the interval [a,b], then with the above notation the *definite integral of f(x) between a and b* is then defined as:

$$\int_a^b f(x)dx = \lim_{n\to\infty} [f(x_1)\Delta x + f(x_2)\Delta x + ...f(x_n)\Delta x] = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x \tag{3}$$

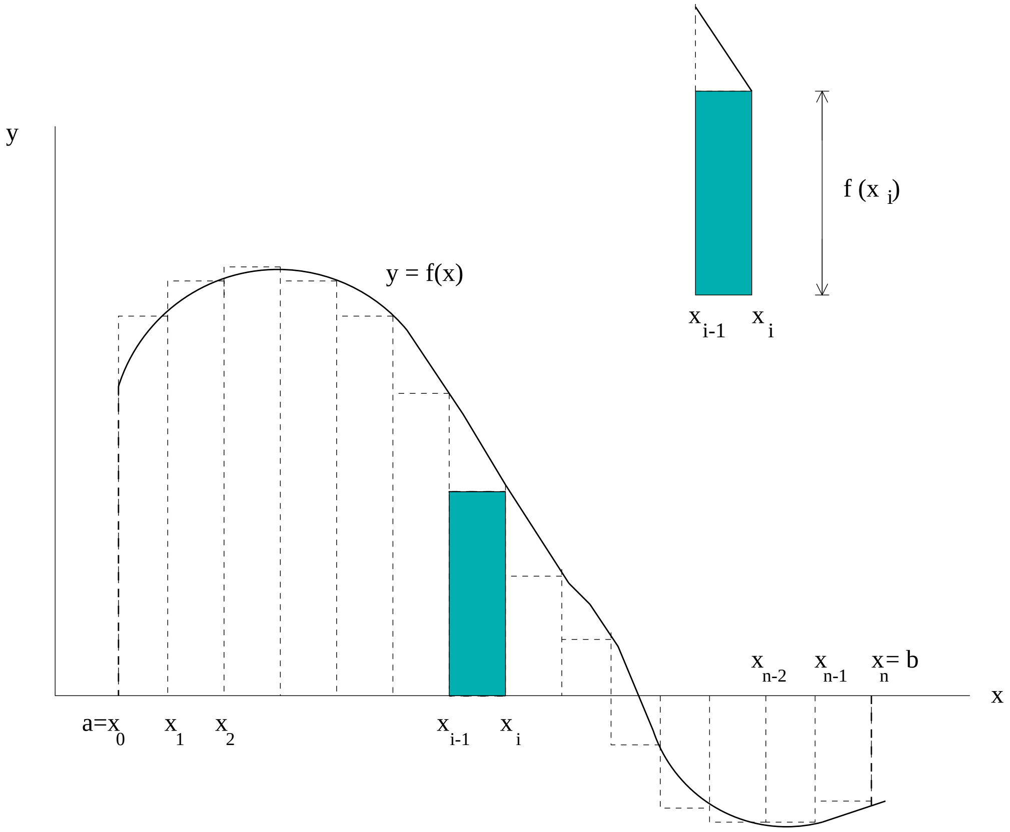

Figure 1:

Of course, calculating such a limit may be quite difficult, luckily there is an easy way of calculating the definite integral of a function if we know an antiderivative of the function.The following theorem is called the fundamental theorem of calculus,it gives an easy way of calculating definite integrals and should be studied carefully. its ramifications shall be felt throughout the remainder of the course.

Theorem 4.2. The Fundamental Theorem of Calculus
Suppose $f(x)$ is continuous on the interval [a,b], and that $F(x)$ is an antiderivative of $f(x)$ on [a,b]. Then:

$$\int_a^b f(x)dx = F(b) - F(a)$$

or equivalently

$$\int_a^b F'(x)dx = F(b) - F(a)$$

Definition 4.3. The numbers a, and b above are called the **limits of integration**.

In the following example watch for the new use of square brackets.

Example 1. Find $\int\limits_1^2 x^3 dx$

Solution:

$\int\limits_1^2 x^3 dx \quad = [\frac{1}{4}x^4]_1^2$ (new use of] ,$\frac{1}{4}x^4$ is an antiderivative of x^3)

$= \frac{1}{4}(2)^4 - \frac{1}{4}(1)^4$

$= 4 - \frac{1}{4} = \frac{15}{4}$

Theorem 4.4. *If $f(x)$ is a positive valued function between $x = a$ and $x = b$, then the area between the graph of $f(x)$ and the x axis is given by* $\int_a^b f(x)dx$

The proof behind the above theorem is more profound than the material for this course, but we can look at the idea behind it. The integral sign $\int$ is shaped like an “S” for a good reason, it represents the “sum” in the above theorem, the integral is used to sum up areas of extremely thin rectangles to give the entire area required. As we let the width Δx get smaller and smaller, the sections begin to look more and more like rectangles. Taking limits as Δx approaches 0, we rename Δx, dx, and the height of the rectangle at x is simply $f(x)$ since f is positive, which makes the area of the rectangle $f(x)dx$. We then integrate between a and b to add up all of the areas of these extremely thin rectangles as x varies between a and b. Loosely speaking we translate

$$\int_a^b f(x)dx$$

As the sum, as x goes from a to b, of $f(x) \times dx$

This being said, let’s try some examples.

Example 2. Solve $\int\limits_{-2}^{4} \frac{-x}{2} dx$

Solution: We have

$$\begin{aligned}\int\limits_{-2}^{4} \frac{-x}{2} dx &= \frac{-1}{2} \int\limits_{-2}^{4} x dx \\ &= \frac{-1}{2} \left[\frac{1}{2}x^2\right]_{-2}^{4} \ (\tfrac{1}{2}x^2 \text{ is an antiderivative of } x) \\ &= -\frac{1}{2}[8-2] \\ &= -3\end{aligned}$$

Example 3. Solve $\int\limits_{0}^{2} 3e^x dx$

Solution: We have:

$$\begin{aligned}\int\limits_{0}^{2} 3e^x dx &= 3\int\limits_{0}^{2} e^x dx \\ &= 3[e^x]_0^2 \\ &= 3[e^2 - e^0] \\ &= 3e^2 - 3\end{aligned}$$

Theorem 4.5. *We now list some important properties of definite integrals, these should be studied carefully.*

(1) $\boxed{\int_a^b cf(x)dx = c\int_a^b f(x)dx}$

(2) $\boxed{\int_a^b (f(x)+g(x))dx = \int_a^b f(x)dx + \int_a^b g(x)dx}$

(3) $\boxed{\int_a^b f(x)dx = \int_a^t f(x)dx + \int_t^b f(x)dx}$

(4) $\boxed{\int_a^b f(x)dx = -\int_b^a f(x)dx}$

Example 4. Find $\int_{-2}^{4} f(x)dx$ where $f(x) = \begin{cases} -2x & -2 \leq x \leq 0 \\ x^2 - 2 & 0 < x \leq 2 \\ -2x + 8 & 2 < x \leq 4 \end{cases}$

Solution: Looking at the above rules (number 3 in particular) we see that

$$\begin{aligned} \int_{-2}^{4} f(x)dx \quad &= \int_{-2}^{0} f(x)dx + \int_{0}^{2} f(x)dx + \int_{2}^{4} f(x)dx \\ \text{(by the definition of f(x))} \quad &= \int_{-2}^{0} -2xdx + \int_{0}^{2} (x^2 - 2)dx + \int_{2}^{4} (-2x + 8)dx \\ &= [-x^2]_{-2}^{0} + \left[\tfrac{1}{3}(x^3) - 2x\right]_{0}^{2} + [-x^2 + 8x]_{2}^{4} \\ &= [0 - 4] + [(\tfrac{1}{3}(8) - 4) - (0)] + [(-16 + 32) - (-4 + 16)] \\ &= \tfrac{20}{3} \end{aligned}$$

Another use of the definite integral is to give us the average value of a function (f_{av})over a specific interval.

Theorem 4.6. *The average value of a function $f(x)$ on an interval [a,b] is given by:*

$$\boxed{f_{av} = \tfrac{1}{b-a} \int_{a}^{b} f(x)dx}$$

Example 5. Find the average value of $f(x) = -x^2 + 4x + 5$ on the interval [1,3].
Solution: By the above theorem, with $a = 1$ and $b = 3$ we have:

$$\begin{aligned} f_{av} \quad &= \tfrac{1}{b-a} \int_{a}^{b} f(x)dx \\ &= \tfrac{1}{3-1} \int_{1}^{3} (-x^2 + 4x + 5) \\ &= \tfrac{1}{2} \left(\tfrac{-1}{3} x^3 + 2x^2 + 5x \right]_{1}^{2} \\ &= \tfrac{1}{2}[(-9 + 18 + 15) - (-\tfrac{1}{3} + 2 + 5)] \\ &= \tfrac{1}{2}(\tfrac{52}{3}) \\ &= \tfrac{27}{3} \end{aligned}$$

We have now learned the basics of definite integrals. What we will now look at is how to apply the methods we know of to solve integrals (substitution and integration by parts) to definite integrals.

Substitution and Definite Integration

The formal statement of the substitution rule with definite integrals may look a bit cumbersome, but after applying it a few times becomes quite easy. Suppose we have an integral which requires substitution. Then the integrand is going to have the form $f(g(x))dx$ where $g(x)$ is the candidate we select for u. In the original integral we are integrating with respect to x, between say a and b. After substituting $u = g(x)$, our new integrand will be in terms of u and we will be integrating with respect to u. So it makes sense that we must change the limits of integration, (a and b in our case) which are the values that x ranges through in the original integral, to new values which give the range of the variable u. Formally:

$$\boxed{\int_a^b f(g(x))dx = \int_{g(a)}^{g(b)} f(u)du \left(= \int_{u(a)}^{u(b)} f(u)du\right)}$$

Example 6. Solve by using substitution: $\int_0^1 (2x+1)^2 dx$

Solution: we proceed as usual with the substitution, the only variation being that we must find the "new" limits of integration (which will depend on our choice of u). Using our rules of thumb for substitution, or plain old experience we choose:

$$\boxed{u = (2x+1)} \Rightarrow \frac{du}{dx} = 2 \Rightarrow \boxed{dx = \frac{du}{2}}$$

We now solve for the new limits of integration, with this choice of u we see that

$$x = 1 \Rightarrow u = (2(1)+1) = 3$$

and

$$x = 0 \Rightarrow u = (2(0)+1) = 1$$

i.e. $u(1) = 3$, and $u(0) = 1$ so our new integral is:

$$\begin{aligned} \textstyle\int_1^3 u^5 \frac{du}{2} &= \textstyle\frac{1}{2} \int_1^3 u^5 du \\ &= \textstyle\frac{1}{2} \left(\frac{u^6}{6} \right]_1^3 \\ &= \textstyle\frac{1}{2} \left(\frac{3^6}{6} - \frac{1^6}{6} \right) \\ &= \textstyle\frac{3^5}{4} - \frac{1}{12} \end{aligned}$$

Example 7. Solve by using substitution:

$$\int_{-1}^{3} (x^2+1)^3 2x dx$$

Solution: Using our rules of thumb for substitution, or plain old experience we choose:

$$\boxed{u = (x^2+1)} \Rightarrow \frac{du}{dx} = 2x \Rightarrow \boxed{dx = \frac{du}{2x}}$$

We now solve for the new limits of integration, with this choice of u we see that

$$x = 3 \Rightarrow u = (3^2+1) = 10$$

and

$$x = -1 \Rightarrow u = (-1)^2 + 1) = 2$$

i.e. $u(3) = 10$, and $u(-1) = 2$ so our new integral is:

$$\begin{aligned} \textstyle\int_2^{10} u^3 2x \frac{du}{2x} &= \textstyle\int_2^{10} u^3 du \\ &= \textstyle\left(\frac{u^4}{4} \right]_2^{10} \\ &= \textstyle\left(\frac{10^4}{4} - \frac{4^4}{4} \right) \\ &= 2500 - 4 = 2496 \end{aligned}$$

Integration By Parts and The Definite Integral

The integration by parts formula generalizes in the natural way to definite integrals. Formally stated, the I.B.P. formula is:

$$\boxed{\int_a^b udv = [uv]_a^b - \int_a^b vdu}$$

Notice that since integration by parts does not actually change the variable of integration (if the original integral was in terms of x, then after using I.B.P. the solution is still interms of x) it is not necessary to change the limits of integration like in substitution. We will do a couple of examples to illustrate.

Example 8. Solve: $\int_1^2 xln(x)dx$

Solution: As always we attempt to use substitution first and it will not work here (verify for yourself), so we turn to I.B.P.

Letting $u = ln(x)$ and $dv = xdx$, we get:

$$u = ln(x) \Rightarrow du = \frac{1}{x}dx$$

$$dv = xdx \Rightarrow v = \frac{x^2}{2}$$

Substituting into the I.B.P. formula

$$\int_a^b udv = [uv]_a^b - \int_a^b vdu$$

we get:

$$\begin{aligned} \int_1^2 xln(x)dx &= [\tfrac{x^2}{2}ln(x)]_1^2 - \int_1^2 \tfrac{x^2}{2}\tfrac{1}{x}dx \\ &= [\tfrac{x^2}{2}ln(x)]_1^2 - \int_1^2 \tfrac{x}{2}dx \\ &= [\tfrac{x^2}{2}ln(x)]_1^2 - [\tfrac{x^2}{4}]_1^2 \\ &= (2ln(2) - 0) - \left(\tfrac{4}{4} - \tfrac{1}{4}\right) \\ &= 2ln(2) - \tfrac{3}{4} \approx 0.6363 \end{aligned}$$

Example 9. Solve:

$$\int_0^3 xe^{3x}dx$$

Solution: As always we attempt to use substitution first and it will not work here (verify for yourself), so we turn to I.B.P.
Letting $u = x$ and $dv = e^{3x}dx$, we get:

$$u = x \Rightarrow \frac{du}{dx} = 1 \Rightarrow du = dx$$

and

$$dv = e^{3x}dx \Rightarrow v = \frac{e^3x}{3}$$

Substituting into the I.B.P. formula:

$$\int_a^b udv = [uv]_a^b - \int_a^b vdu$$

we get:

$$\begin{aligned}
\textstyle\int_0^3 xe^{3x}dx &= [x\tfrac{e^3x}{3}]_0^3 - \textstyle\int_0^3 \tfrac{e^{3x}}{3}dx \\
&= [\tfrac{xe^{3x}}{3}]_0^3 - \textstyle\int_0^3 \tfrac{e^{3x}}{3}dx \\
&= [\tfrac{xe^{3x}}{3}]_0^3 - [\tfrac{e^{3x}}{9}]_0^3 \\
&= (e^9 - 0) - \left(\tfrac{e^9}{9} - \tfrac{1}{9}\right) \\
&= \tfrac{8e^9+1}{9}
\end{aligned}$$